

The Strong-Coupling Expansion in Simplicial Quantum Gravity

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We construct the strong-coupling series in 4d simplicial quantum gravity up to volume 38. It is used to calculate estimates for the string susceptibility exponent γ for various modifications of the theory. It provides a very efficient way to get a first view of the phase structure of the models.

1. INTRODUCTION

Euclidean quantum gravity is formally defined by the partition function

$$Z = \int \frac{D[g_{\mu\nu}]}{\text{Vol}(\text{diff})} e^{-S[g_{\mu\nu}]} \quad (1)$$

where the Einstein-Hilbert action

$$S[g_{\mu\nu}] = \int d^4\xi \sqrt{g(\xi)} \left\{ \lambda - \frac{1}{16\pi G} R(\xi) \right\}. \quad (2)$$

Discretizing the theory on an ensemble of four-dimensional manifolds T , consisting of N_4 4-simplexes glued together along the faces, and assuming a metric where all link lengths are equal to one, leads to simplicial quantum gravity:

$$Z_{GC}(\kappa_2, \kappa_4) = \sum_{\{T\}} \frac{1}{C(T)} e^{\kappa_2 N_2(T) - \kappa_4 N_4(T)}. \quad (3)$$

Here κ_4 corresponds to the cosmological constant λ , κ_2 to the inverse Newton's constant G , and N_2 is the number of 2-simplexes (triangles). $C(T)$ is the symmetry factor of triangulation T , the number of equivalent labelings of the vertexes. Eq. (3) is related to the canonical (fixed volume) partition function Z_{N_4} through

$$Z_{GC}(\kappa_2, \kappa_4) = \sum_{N_4} e^{-\kappa_4 N_4} Z_{N_4}(\kappa_2). \quad (4)$$

Although it is not proven analytically, one assumes that Z_{N_4} is exponentially bounded in N_4 . Moreover, in our analysis we assume that

$$Z_{N_4}(\kappa_2) \sim N_4^{\gamma(\kappa_2)-3} e^{\mu_c(\kappa_2)N_4} (1 + \frac{c_1}{N_4} + \dots) \quad (5)$$

when $N_4 \rightarrow \infty$, where γ determines the critical behavior of the grand-canonical partition function: $Z_{GC} \sim (\mu_c - \kappa_4)^{2-\gamma}$ as $\kappa_4 \rightarrow \mu_c$.

Simulating this model one observes two phases; a branched polymer phase for $\kappa_2 \gtrsim 1.3$, and a crumpled phase for $\kappa_2 \lesssim 1.3$. In the branched polymer phase $\gamma = \frac{1}{2}$, whereas in the crumpled phase the sub-leading correction is exponential, rather than Eq. (5); this corresponding formally to $\gamma = -\infty$. Separating the two phases is a first order phase transition, hence no interesting critical behavior is observed. For that one may need to modify the simple action in Eq. (3) above. However, as investigating every modification requires extensive numerical simulations, it is important to have some analytical guidance. One such is the strong-coupling expansion.

We have used this expansion to investigate two modifications of the model, described in Ref. [2], namely the interaction of gravity with gauge matter fields, and a modified measure.

2. STRONG-COUPING EXPANSION

The strong-coupling expansion consist of explicit construction of the smallest configurations and their symmetry factors, and a calculation of the additional weights corresponding to a given model. In two dimensions matrix models allow a recursive construction of the series; in four dimensions, though, such methods are not available, but an alternative procedure based on numerical simulations has been developed [2]:

(a) Using Monte Carlo simulations we identify all *distinct* triangulations, up to a given volume. To identify the triangulations, we introduce a hash function $f(T)$; this function has to be complicated enough to distinguish between combinatorially different triangulations, while simple enough for a repeated calculation in the MC simulations.

Only triangulations that can not be reduced by applying one of the volume decreasing geometric moves have to be considered in the MC. All others are systematically constructed from smaller ones. This reduces the MC efforts substantially as, in practice, 99.9% of the triangulations turn out to be reducible.

(b) For each triangulation we calculate the corresponding symmetry factor $C(T)$ by comparing all possible permutations of the vertex labels (only permutations of vertexes of same order have to be considered). Comparing the $C(T)$'s to the relative frequency with which different triangulations appear in the MC simulations provides a check on the correct identification of the triangulations.

Using this procedure we have identified all four-dimensional triangulations up to volume $N_4 = 38$, all in all 1.477.713 distinct triangulations. This gives the first 16 terms in the strong-coupling series, however, due to different finite-size corrections these terms split into two distinct series which have to be analyzed separately.

The strong-coupling series is analyzed using an appropriate series extrapolation method, e.g. the *ratio method*. Assuming the asymptotic behavior Eq. (5) this yields the critical coupling μ_c and the exponent γ .

3. RESULTS

Using the strong-coupling expansion it is easy to explore a large set of modified models of 4d simplicial gravity. Here we will give results just for two such models:

(a) Modified measure:

$$M(T) \sim \prod_{j=1}^{N_2} o(t_j)^\beta \quad (6)$$

where $o(t_j)$ is the order of the j 'th triangle, i.e. to how many 4-simplexes it is attached.

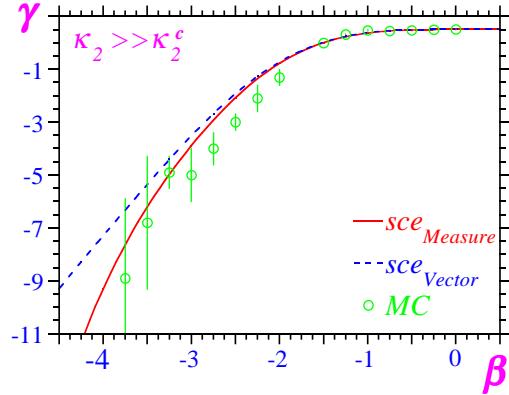


Figure 1. Strong-coupling estimates of γ for both a modified measure (solid line) and f copies of vector fields (dashed line). The circles are values of γ from MC with modified measure.

(b) f copies of vector fields:

$$Z_V(T) = \int' \prod_{l \in T} dA(l) e^{-S[A(l)]} \quad (7)$$

$$S(A(l)) = \sum_{t_{abc}} o(t_{abc}) [A(l_{ab}) + A(l_{bc}) + A(l_{ca})]^2$$

Here $A(l)$ are non-compact $U(1)$ gauge fields residing on the links and the sum is over all triangles. The prime indicates that the zero modes of the gauge field are not integrated over.

The weights corresponding to both models are calculated explicitly for every configuration T . The series so obtained are analyzed with the ratio method, assuming the asymptotic behavior Eq. (5). In Fig. 1 we shown an example of γ , for $\kappa_2 \geq \kappa_2^c$, for both the models as we vary β and f respectively. For consistency we have verified that successive approximations in the analysis of the series converge, what is plotted is the result of the highest approximation. These results are in excellent agreement with values of γ measured in MC simulations of Eq. (6) for $N_4 = 4000$. Furthermore, the two models yield the same variation of γ provided one takes

$$\beta = -\frac{f}{2} - \frac{1}{4}. \quad (8)$$

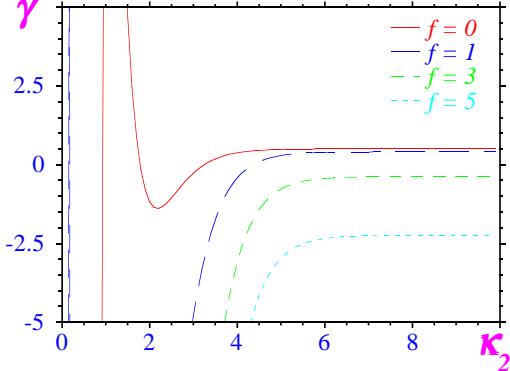


Figure 2. Variations in γ versus κ_2 for f copies of vector fields coupled to 4d-gravity.

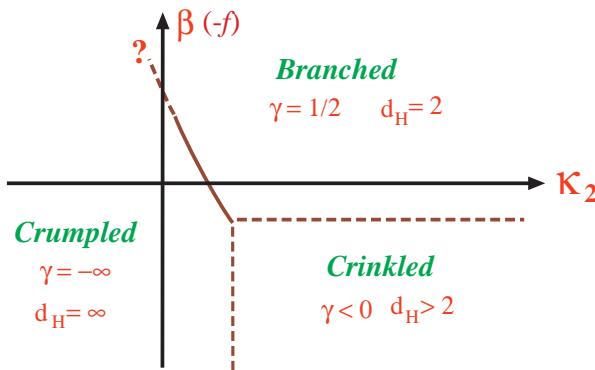


Figure 3. A schematic phase diagram of simplicial gravity coupled to either a measure term, or to vector fields.

This universality appears in a variety of “reasonable” modifications of the model Eq. (3), all of which give the same phase structure (modulo a trivial re-scaling of parameters) [2].

Fig. 2 shows γ as we vary κ_2 , for different number of vector fields, given by the strong-coupling series. The values of γ become unstable for small κ_2 ; this is consistent with the fact that in the corresponding MC simulations we observe a phase transition to a crumpled phase. This transition is evident in the series extrapolation as successive approximations fail to converge both in the crumpled phase and in the critical region.

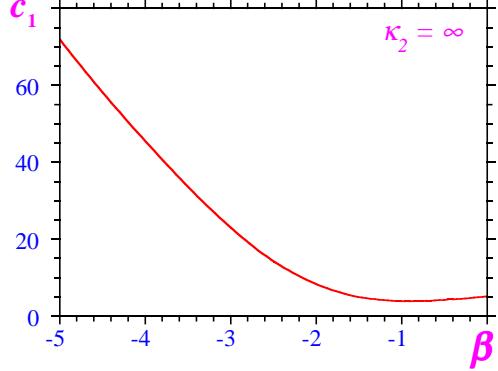


Figure 4. The first corrections term in Eq. (5).

From these and further investigations we conclude that phase diagrams of the two models are quite similar, as depicted schematically in Fig. 3. In addition to the crumpled and branched polymer phases, a new *crinkled* phase appears, for sufficiently strong coupling to either matter or the measure term, characterized by a finite $\gamma < 0$.

As $\kappa_2 \rightarrow \infty$, the finite-size effects become, rather surprisingly, less important in the pure gravity model. This is related to the dominance of a particular class of manifolds, *stacked spheres*, in this limit [3]. However, as a coupling to a measure term is added the finite-size effects increase again, as shown in Fig. 4 were we plot the the first correction term to Eq. (5) for $\kappa_2 = \infty$ and varying β . The increased finite-size correction coincided with the appearance of the crinkled phase.

The examples given above show that the strong-coupling series gives a good qualitative view of the phase structure of modified models of 4d simplicial gravity. It provides us with a powerful tool and should be used before starting extensive computer simulations of a particular model.

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